Philadelphia University

Lecture Notes for 650364

Probability & Random Variables

Lecture 7: Multiple Random Variables

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- **Discrete Case**: If X and Y are two discrete random variables, we define the ***joint probability function*** of X and Y by

$$P(X = x, Y = y) = f(x, y)$$

Where

$$1.\ f(x, y) \geq 0$$
$$2.\ \sum_x \sum_y f(x, y) = 1$$

- Suppose that X can assume any one of m values $x_1, x_2, \ldots, x_m$ and Y can assume any one of n values $y_1, y_2, \ldots, y_n$. Then the probability of the event that $X = x_j$ and $Y = y_k$ is given by

$$P(X = x_j, Y = y_k) = f(x_j, y_k)$$

- A **joint probability function** for X and Y can be represented by a ***joint probability table***
- The probability that $X = x_j$ is obtained by adding all entries in the row corresponding to x_i and is given by

$$P(X = x_j) = f_1(x_j) = \sum_{k=1}^{n} f(x_j, y_k)$$

Y \ X	y_1	y_2	$\cdots$	y_n	Totals ↓
x_1	$f(x_1, y_1)$	$f(x_1, y_2)$	$\cdots$	$f(x_1, y_n)$	$f_1(x_1)$
x_2	$f(x_2, y_1)$	$f(x_2, y_2)$	$\cdots$	$f(x_2, y_n)$	$f_1(x_2)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_m	$f(x_m, y_1)$	$f(x_m, y_2)$	$\cdots$	$f(x_m, y_n)$	$f_1(x_m)$
Totals →	$f_2(y_1)$	$f_2(y_2)$	$\cdots$	$f_2(y_n)$	1 ← Grand Total

- ✓ Similarly the probability that $Y = y_k$ is obtained by adding all entries in the column corresponding to y_k and is given by

$$P(Y = y_k) = f_2(y_k) = \sum_{j=1}^{m} f(x_j, y_k)$$

- ✓ We often refer to $f_1(x_j)$ and $f_2(y_k)$ [or simply $f_1(x)$ and $f_2(y)$]as the ***marginal probability functions*** of X and Y, respectively
- ✓ It should also be noted that

$$\sum_{j=1}^{m} f_1(x_j) = 1 \qquad \sum_{k=1}^{n} f_2(y_k) = 1$$

Which can be written

$$\sum_{j=1}^{m} \sum_{k=1}^{n} f(x_j, y_k) = 1$$

- ✓ This is simply the statement that the **total probability** of all entries is 1. The ***joint distribution function*** of ***X*** and ***Y*** is defined by

$$F(x, y) = P(X \le x, Y \le y) = \sum_{u \le x} \sum_{v \le y} f(u, v)$$

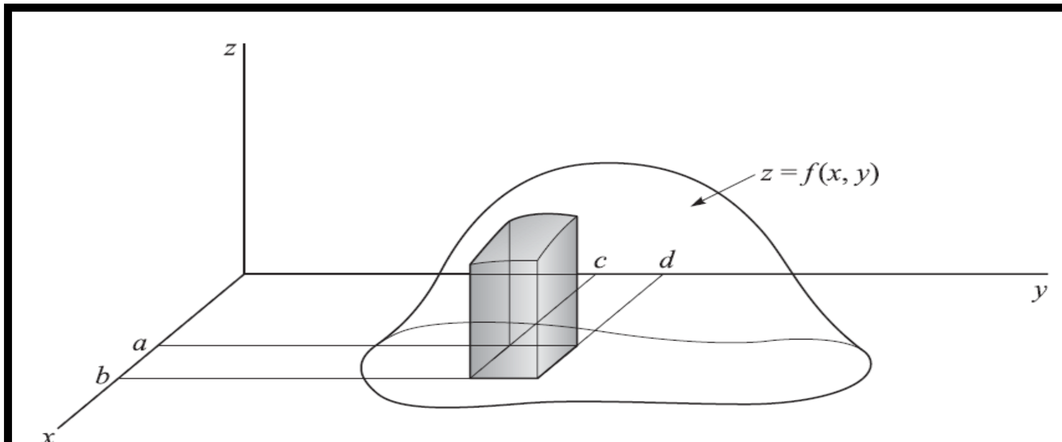
In Table, ***F*(*x*, *y*)** is the sum of all entries for which $x_j \le x$ and $y_k \le y$.

- ✓ **Continuous Case**: the ***joint probability function*** for the random variables ***X*** and ***Y*** (or, as it is more commonly called, the ***joint density function*** of ***X*** and ***Y***) is defined by

1. $f(x, y) \ge 0$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$

- ✓ Graphically ***z* =*f* (*x*, *y*)** represents a surface, called the ***probability surface***
- ✓ The probability that ***X*** lies between ***a*** and ***b*** while ***Y*** lies between ***c*** and ***d*** is given graphically by the shaded volume of Fig. and mathematically by

$$P(a < X < b, c < Y < d) = \int_{x=a}^{b}\int_{y=c}^{d} f(x, y)\,dx\,dy$$



- ✓ The ***joint distribution function*** of ***X*** and ***Y*** in this case is defined by

$$F(x, y) = P(X \le x, Y \le y) = \int_{u=-\infty}^{x}\int_{v=-\infty}^{y} f(u, v)\,du\,dv$$

✓ It follows in **analogy** that

$$\frac{\partial^2 F}{\partial x\, \partial y} = f(x, y)$$

i.e., the **density function** is obtained by **differentiating** the distribution function with respect to ***x*** and ***y***.

✓ The ***marginal distribution functions***, or simply the *distribution functions*, of ***X*** and ***Y***, respectively

$$P(X \le x) = F_1(x) = \int_{u=-\infty}^{x} \int_{v=-\infty}^{\infty} f(u, v)\, du\, dv$$

$$P(Y \le y) = F_2(y) = \int_{u=-\infty}^{\infty} \int_{v=-\infty}^{y} f(u, v)\, du\, dv$$

The **derivatives** of the above equations with respect to ***x*** and ***y*** are then called the ***marginal density functions,*** or simply the ***density functions***, of ***X*** and ***Y*** and are given by

$$f_1(x) = \int_{v=-\infty}^{\infty} f(x, v)\, dv \qquad f_2(y) = \int_{u=-\infty}^{\infty} f(u, y)\, du$$

Independent Random Variables

✓ Suppose that ***X*** and ***Y*** are **discrete random variables**. If the events ***X* =*x*** and ***Y* =*y*** are **independent events** for all ***x*** and ***y***, then we say that ***X*** and ***Y*** are **independent random variables**. In such case,

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

Or

$$f(x, y) = f_1(x)f_2(y)$$

- The **joint probability function** **f (x, y)** can be expressed as the **product** of a function of ***x*** alone and a function of ***y*** alone, ***X*** and ***Y*** are independent.

✓ If ***X*** and ***Y*** are continuous random variables, we say that they are ***independent random variables*** if the events ***X* ≤*x*** and ***Y* ≤*y*** are **independent** events for all ***x*** and ***y***. In such case we can write

$$P(X \le x, Y \le y) = P(X \le x)P(Y \le y)$$

Or

$$F(x, y) = F_1(x)F_2(y)$$

- Where $F_1(x)$ and $F_2(y)$ are the **marginal distribution functions** of X and Y, respectively. If, however, $F(x, y)$ cannot be so expressed as a product, then X and Y are dependent.

Conditional Distributions

- We already know that if $P(A) > 0$,

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

- If X and Y are discrete random variables and we have the events $(A: X = x)$, $(B: Y = y)$, then above equation becomes

$$P(Y = y \mid X = x) = \frac{f(x, y)}{f_1(x)}$$

Where $f(x, y) = P(X = x, Y = y)$ is the **joint probability function** and $f_1(x)$ is the **marginal probability function** for X. We define

$$f(y \mid x) \equiv \frac{f(x, y)}{f_1(x)}$$

and call it the ***conditional probability function*** *of Y given X*.

- Similarly, the **conditional probability function** of X given Y is

$$f(x \mid y) \equiv \frac{f(x, y)}{f_2(y)}$$

✓ These ideas are easily extended to the case where ***X***, ***Y*** continuous random variables are. For example, the ***conditional density function*** *of **Y** given **X*** is

$$f(y \mid x) \equiv \frac{f(x, y)}{f_1(x)}$$

Examples

✓ **Example 1**: The **joint probability function** of two discrete random variables ***X*** and ***Y*** is given by $f(x, y) = c(2x + y)$, where ***x*** and ***y*** can assume all integers such that $0 \leq x \leq 2,\ 0 \leq y \leq 3$, and $f(x, y) = 0$ otherwise.

a) **Find the value of the constant c.**

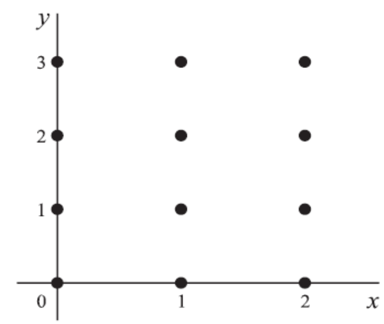
b) **Find $P(X = 2, Y = 1)$.**

c) **Find $P(X \geq 1, Y \leq 2)$.**

- **Solution**

a) The sample points (x, y) for which probabilities are different from zero are indicated in Fig. The probabilities associated with these points, given by $c(2x + y)$, are shown in Table. Since the grand total, $42c$, must equal 1, we have $c = \frac{1}{42}$.

X \ Y	0	1	2	3	Totals ↓
0	0	c	$2c$	$3c$	$6c$
1	$2c$	$3c$	$4c$	$5c$	$14c$
2	$4c$	$5c$	$6c$	$7c$	$22c$
Totals →	$6c$	$9c$	$12c$	$15c$	$42c$



b) From Table

$$P(X = 2, Y = 1) = 5c + \frac{5}{42}$$

c) From Table

$$P(X \geq 1, Y \leq 2) = \sum_{x \geq 1} \sum_{y \leq 2} f(x, y)$$
$$= (2c + 3c + 4c)(4c + 5c + 6c)$$
$$= 24c = \frac{24}{42} = \frac{4}{7}$$

✓ **Example 2**: Find the **marginal probability functions (a)** of ***X*** and **(b)** of ***Y*** for the random variables of **example 1**.

- **Solution**

a) The marginal probability function for ***X*** is given by $P(X = x) = f_1(x)$ and can be obtained from the margin totals in the right-hand column of the table.

$$P(X = x) = f_1(x) = \begin{cases} 6c = 1/7 & x = 0 \\ 14c = 1/3 & x = 1 \\ 22c = 11/21 & x = 2 \end{cases}$$

Check: $\frac{1}{7} + \frac{1}{3} + \frac{11}{21} = 1$

b) The marginal probability function for ***Y*** is given by $P(Y = y) = f_2(y)$ and can be obtained from the margin totals in the last row of the table.

$$P(Y = y) = f_2(y) = \begin{cases} 6c = 1/7 & y = 0 \\ 9c = 3/14 & y = 1 \\ 12c = 2/7 & y = 2 \\ 15c = 5/14 & y = 3 \end{cases}$$

Check: $\frac{1}{7} + \frac{3}{14} + \frac{2}{7} + \frac{5}{14} = 1$

- ✓ **Example 3**: Show that the random variables ***X*** and ***Y*** of **example 1** are **dependent**.
  - **Solution**

If the random variables ***X*** and ***Y*** are independent, then

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

But, as seen from example 1 and example 2

$$P(X = 2, Y = 1) = \frac{5}{42} \qquad P(X = 2) = \frac{11}{21} \qquad P(Y = 1) = \frac{3}{14}$$

So that

$$P(X = 2, Y = 1) \neq P(X = 2)P(Y = 1)$$

The result also follows from the fact that the joint probability function $(2x + y)/42$ cannot be expressed as a function of x alone times a function of y alone.

Example 4: The **joint density function** of two continuous random variables X and Y is

$$f(x, y) = \begin{cases} cxy & 0 < x < 4, 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

a) **Find the value of the constant c.**

b) **Find $P(1 < X < 2, 2 < Y < 3)$.**

c) **Find $P(X \geq 3, Y \leq 2)$.**

- **Solution**

a) We must have the total probability equal to 1, i.e.,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$

Using the definition of $f(x, y)$, the integral has the value

$$\int_{x=0}^{4}\int_{y=1}^{5} cxy\,dx\,dy = c\int_{x=0}^{4}\left[\int_{y=1}^{5} xy\,dy\right]dx$$

$$= c\int_{z=0}^{4} \frac{xy^2}{2}\bigg|_{y=1}^{5} dx = c\int_{x=0}^{4}\left(\frac{25x}{2} - \frac{x}{2}\right)dx$$

$$= c\int_{x=0}^{4} 12x\,dx = c(6x^2)\bigg|_{x=0}^{4} = 96c$$

Then $96c = 1$ and $c = 1/96$.

(b) Using the value of c found in (a), we have

$$P(1 < X < 2, 2 < Y < 3) = \int_{x=1}^{2}\int_{y=2}^{3} \frac{xy}{96}\,dx\,dy$$

$$= \frac{1}{96}\int_{x=1}^{2}\left[\int_{y=2}^{3} xy\,dy\right]dx = \frac{1}{96}\int_{x=1}^{2} \frac{xy^2}{2}\bigg|_{y=2}^{3} dx$$

$$= \frac{1}{96}\int_{x=1}^{2} \frac{5x}{2}dx = \frac{5}{192}\left(\frac{x^2}{2}\right)\bigg|_{1}^{2} = \frac{5}{128}$$

(c)
$$P(X \geq 3, Y \leq 2) = \int_{x=3}^{4}\int_{y=1}^{2} \frac{xy}{96}\,dx\,dy$$
$$= \frac{1}{96}\int_{x=3}^{4}\left[\int_{y=1}^{2} xy\,dy\right]dx = \frac{1}{96}\int_{x=3}^{4} \left.\frac{xy^2}{2}\right|_{y=1}^{2} dx$$
$$= \frac{1}{96}\int_{x=3}^{4} \frac{3x}{2}\,dx = \frac{7}{128}$$

✓ **Example 5**: Find the **marginal distribution functions** **(a)** of ***X*** and **(b)** of ***Y*** for **example 4**.

- **Solution**
  - **a)** The marginal distribution function for ***X*** if $0 \leq x < 4$ is

$$F_1(x) = P(X \leq x) = \int_{u=-\infty}^{x}\int_{v=-\infty}^{\infty} f(u, v)\,du\,dv$$
$$= \int_{u=0}^{x}\int_{v=1}^{5} \frac{uv}{96}\,du\,dv$$
$$= \frac{1}{96}\int_{u=0}^{x}\left[\int_{v=1}^{5} uv\,dv\right]du = \frac{x^2}{16}$$

For $x \geq 4$, $F_1(x) = 1$; for $x < 0$, $F_1(x) = 0$. Thus

$$F_1(x) = \begin{cases} 0 & x < 0 \\ x^2/16 & 0 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

b) The marginal distribution function for Y if $1 \leq y < 5$ is

$$F_2(y) = P(Y \leq y) = \int_{u=-\infty}^{\infty} \int_{v=1}^{y} f(u, v)\, du\, dv$$

$$= \int_{u=0}^{4} \int_{v=1}^{y} \frac{uv}{96}\, du\, dv = \frac{y^2 - 1}{24}$$

For $y \geq 5$, $F_2(y) = 1$. For $y < 1$, $F_2(y) = 0$. Thus

$$F_2(y) = \begin{cases} 0 & y < 1 \\ (y^2 - 1)/24 & 1 \leq y < 5 \\ 1 & y \geq 5 \end{cases}$$

✓ **Example 6**: Find (a) $f(y|2)$, (b) $P(Y = 1|X = 2)$ for the distribution of **example 1**

- **Solution:** Using the result in example 1 and example 2, we have:

(a)

$$f(y|x) = \frac{f(x, y)}{f_1(x)} = \frac{(2x + y)/42}{f_1(x)}$$

so that with $x = 2$

$$f(y|2) = \frac{(4 + y)/42}{11/21} = \frac{4 + y}{22}$$

(b)
$$P(Y = 1 | X = 2) = f(1|2) = \frac{5}{22}$$

✓ **Example 7**: If X and Y have the joint density function

$$f(x, y) = \begin{cases} \frac{3}{4} + xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find **(a)** $f(y|x)$, **(b)** $P(Y > \frac{1}{2} | \frac{1}{2} < X < \frac{1}{2} + dx)$

- **Solution:**

(a) For $0 < x < 1$,

$$f_1(x) = \int_0^1 \left(\frac{3}{4} + xy\right) dy = \frac{3}{4} + \frac{x}{2}$$

and

$$f(y|x) = \frac{f(x, y)}{f_1(x)} = \begin{cases} \dfrac{3 + 4xy}{3 + 2x} & 0 < y < 1 \\ 0 & \text{other } y \end{cases}$$

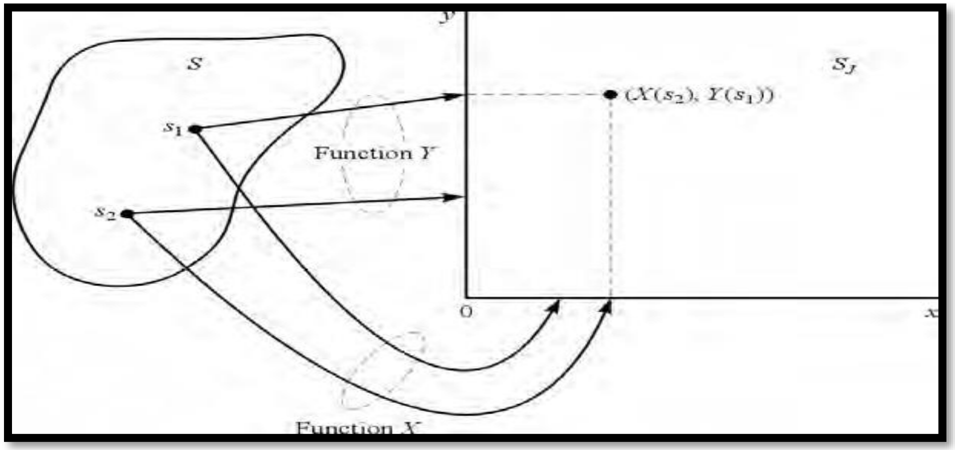
For other values of x, $f(y|x)$ is not defined.

(b)

$$P(Y > \tfrac{1}{2} | \tfrac{1}{2} < X < \tfrac{1}{2} + dx) = \int_{1/2}^{\infty} f(y | \tfrac{1}{2}) dy = \int_{1/2}^{1} \frac{3 + 2y}{4} dy = \frac{9}{16}$$

Vector Random Variables

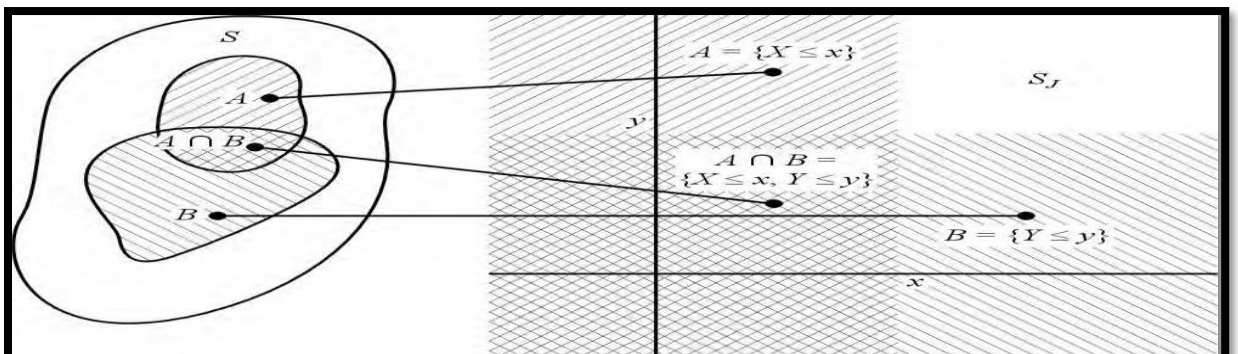
- ✓ Let **X** and **Y** denote two random variables defined on a sample space **S**, where specific values of **X** and **Y** are denoted by **x** and **y**. Then any ordered pair of numbers **(x, y)** may be considered a **random point** in **xy** plane. The point may be taken a specific value of a **vector random variable**.
- ✓ The plane of all points **(x, y)** in the ranges of **X** and **Y** may be considered a new sample space called a **joint sample space** S_J.



✓ As in the case of one random variable, let us define events **A** and **B** by

$$A = \{X \le x\} \qquad \text{and} \qquad B = \{Y \le y\}$$

✓ The event **A∩B** defined on **S** corresponds to the joint event {**X≤ x** and **Y ≤ y}** defined on $\mathbf{S_J}$.



Joint Distribution and its Properties

- The probabilities of the two events **A={X ≤ x}** and **B={Y ≤ y}** have distribution functions:

$$F_X(x) = P\{X \le x\} \quad \text{and} \quad F_Y(y) = P\{Y \le y\}$$

- We define the probability of the joint event **{X≤ x and Y ≤ y}** by a joint probability distribution function

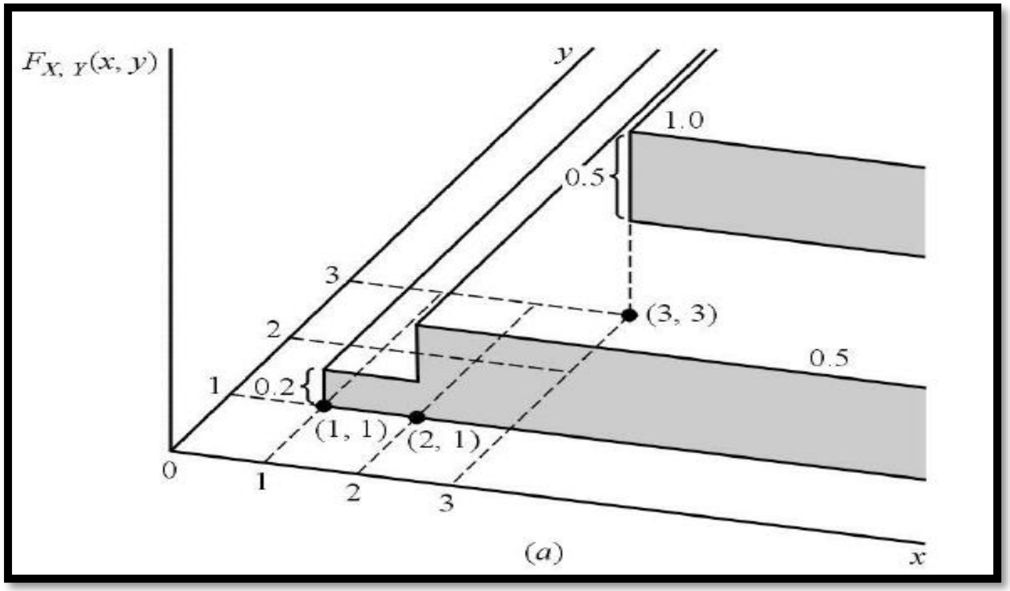
$$F_{X,Y}(x,y) = P\{X \le x, Y \le y\}$$

- It should be clear that

$$P\{X \le x, Y \le y\} = P(A \cap B)$$

- Example: Assume that the joint sample space S_J has only three possible elements **(1,1), (2,1),** and **(3,3).** The probabilities of these elements are to be **P(1,1)=0.2**, **P(2,1)=0.3**, and **P(3,3)=0.5**
  - The distribution function:

$$F_{X,Y}(x,y) = 0.2u(x-1)u(y-1) + 0.3u(x-2)u(y-1) + 0.5u(x-3)u(y-3)$$



✓ **Joint Distribution for Discrete Random Variables**:

- The joint distribution function of discrete random variables **X** and **Y** is given by:

$$F_{X,Y}(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{M} P(x_n, y_m)u(x - x_n)u(y - y_m)$$

- **Example**:

$$F_{X,Y}(x,y) = 0.2u(x-1)u(y-1) + 0.3u(x-2)u(y-1) + 0.5u(x-3)u(y-3)$$

✓ Properties of the Joint Distribution:

1. $F_{X,Y}(-\infty,-\infty)=0, \quad F_{X,Y}(-\infty,y)=0, \quad F_{X,Y}(x,-\infty)=0$
2. $F_{X,Y}(\infty,\infty)=1$
3. $0 \le F_{X,Y}(x,y) \le 1$
4. $F_{X,Y}(x,y)$ is a nondecreasing function
5. $P\{x_1 < X \le x_2, y_1 < Y \le y_2\} = F_{X,Y}(x_2,y_2)+F_{X,Y}(x_1,y_1) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2)$
6. $F_{X,Y}(x,\infty)=F_X(x)$ and $F_{X,Y}(\infty,y)=F_Y(y)$

✓ Marginal Distribution Functions:

- Property 6 above states that the **marginal distribution functions** obtained by

$$F_X(x)=F_{X,Y}(x,\infty) \quad \text{and} \quad F_Y(y)=F_{X,Y}(\infty,y)$$

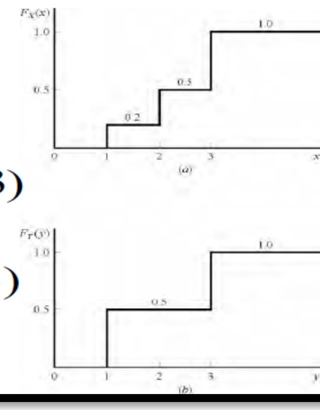
- **Example**:

$$F_{X,Y}(x,y) = 0.2u(x-1)u(y-1) + 0.3u(x-2)u(y-1) + 0.5u(x-3)u(y-3)$$

the marginal distribution functions:

$$F_X(x) = 0.2u(x-1) + 0.3u(x-2) + 0.5u(x-3)$$

$$F_Y(y) = 0.2u(y-1) + 0.3u(y-1) + 0.5u(y-3) = 0.5u(y-1) + 0.5u(y-3)$$



Joint Density and its Properties

- ✓ The **joint probability density** is defined by the **second derivative** of the joint distribution function:

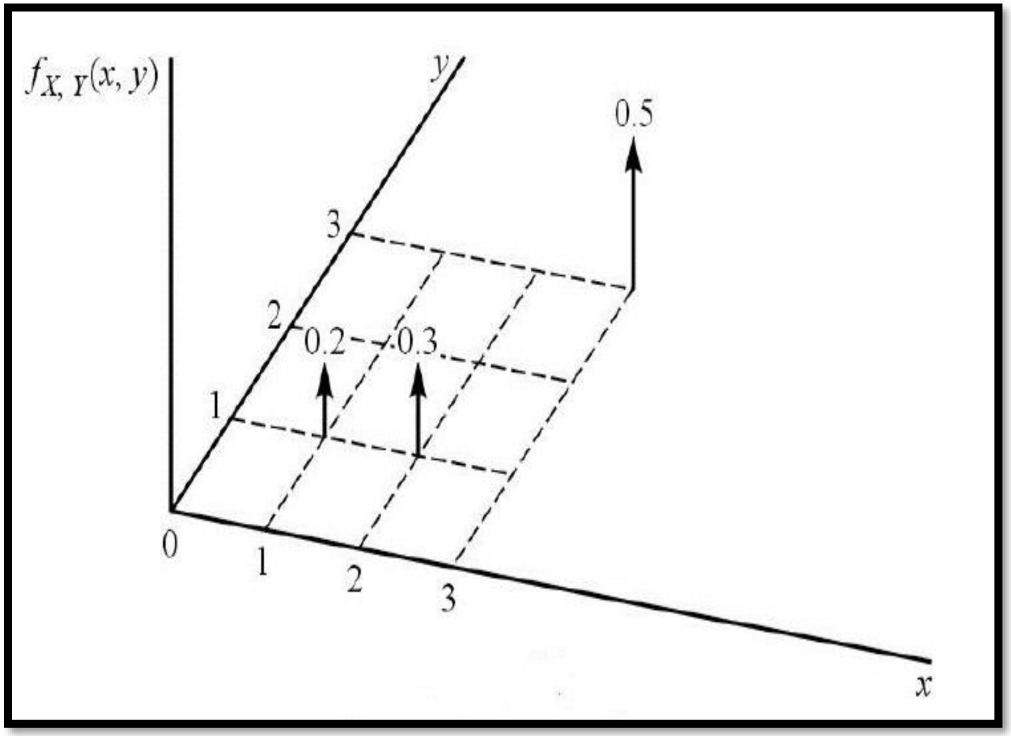
$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

- ✓ Joint Density for Discrete Random Variables: The joint density function of discrete random variables **X** and **Y** is given by:

$$f_{X,Y}(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{M} P(x_n, y_m)\,\delta(x-x_n)\delta(y-y_m)$$

- Example: Assume that the joint sample space S_J has only three possible elements **(1,1), (2,1),** and **(3,3).** The probabilities of these elements are to be **P(1,1)=0.2, P(2,1)=0.3**, and **P(3,3)=0.5**
  - The **density function**:

$$f_{X,Y}(x,y) = 0.2\delta(x-1)\delta(y-1) + 0.3\delta(x-2)\delta(y-1) + 0.5\delta(x-3)\delta(y-3)$$



✓ Properties of the Joint Density:

1. $f_{X,Y}(x,y) \geq 0$

2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$

3. $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\xi_1,\xi_2)d\xi_1 d\xi_2$

4. $F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_2 d\xi_1$

$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_1 d\xi_2$

→ Marginal Distribution Functions

5. $P\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(x,y)dxdy$

6. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$

→ Marginal Density Functions

✓ Marginal Density Functions:

- Property 6 above states that the marginal distribution functions obtained by

$$f_X(x) = \frac{dF_X(x)}{dx} \quad \text{and} \quad f_Y(y) = \frac{dF_Y(y)}{dy}$$

- **Example**:

$$F_{X,Y}(x,y) = 0.2u(x-1)u(y-1) + 0.3u(x-2)u(y-1) + 0.5u(x-3)u(y-3)$$

  - the marginal density functions:

$$f_X(x) = 0.2\delta(x-1) + 0.3\delta(x-2) + 0.5\delta(x-3)$$

$$f_Y(y) = 0.2\delta(y-1) + 0.3\delta(y-1) + 0.5\delta(y-3) = 0.5\delta(y-1) + 0.5\delta(y-3)$$

- **Example**: Find the value of b so that the following function is a valid joint density function

$$g(x,y) = \begin{cases} be^{-x}\cos(y) & 0 \le x \le 2 \text{ and } 0 \le y \le \pi/2 \\ 0 & \text{all other } x \text{ and } y \end{cases}$$

- **Solution**:

$$\int_0^{\pi/2}\int_0^2 be^{-x}\cos(y)\,dx\,dy = b\int_0^2 e^{-x}dx\int_0^{\pi/2}\cos(y)\,dy$$
$$= b(1-e^{-2}) = 1$$

then

$$b = \frac{1}{1-e^{-2}}$$

- **Example**: Find the marginal density functions when the joint density function is given by

$$f_{X,Y}(x,y) = xe^{-x(y+1)}u(x)u(y)$$

- **Solution**:

$$f_X(x) = \int_0^{\infty} xe^{-x(y+1)}u(x)\,dy = xe^{-x}u(x)\int_0^{\infty} e^{-xy}dy$$
$$= xe^{-x}u(x)(1/x) = e^{-x}u(x)$$

and

$$f_Y(y) = \int_0^{\infty} xe^{-x(y+1)}u(y)\,dx = \frac{1}{(y+1)^2}u(y)$$

Statistical Independence

✓ The two random variables **X** and **Y** are called **statistically independent** if

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$

Or

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

- **Example**: For previous example

$$f_{X,Y}(x,y) = xe^{-x(y+1)}u(x)u(y)$$

$$f_X(x) = e^{-x}u(x) \quad and \quad f_Y(y) = \frac{1}{(y+1)^2}u(y)$$

  - **Solution**:

$$f_X(x)f_Y(y) = \frac{e^{-x}}{(y+1)^2}u(x)u(y) \neq f_{X,Y}(x,y)$$

Therefore, the random variables X and Y are **not independent**.

- **Example**: The joint density of two random variables X and Y is

$$f_{X,Y}(x, y) = \frac{1}{12} e^{-(x/4)-(y/3)} u(x) u(y)$$

Determine if X and Y are independent.

- **Solution**:

$$f_X(x) = \int_0^{\infty} (1/12) e^{-x/4} e^{-y/3} u(x) dy = \frac{1}{4} e^{-x/4} u(x)$$

$$f_Y(y) = \int_0^{\infty} (1/12) e^{-x/4} e^{-y/3} u(y) dx = \frac{1}{3} e^{-y/3} u(y)$$

$$f_X(x) f_Y(y) = f_{X,Y}(x, y)$$

Therefore, the random variables X and Y are independent.

Distribution and Density of a Sum of Random Variables

✓ If **W** be a random variable equal to the sum of two independent random variables **X** and **Y:**

$$W = X + Y$$

Then the density function of **W** is the **convolution** of their density functions

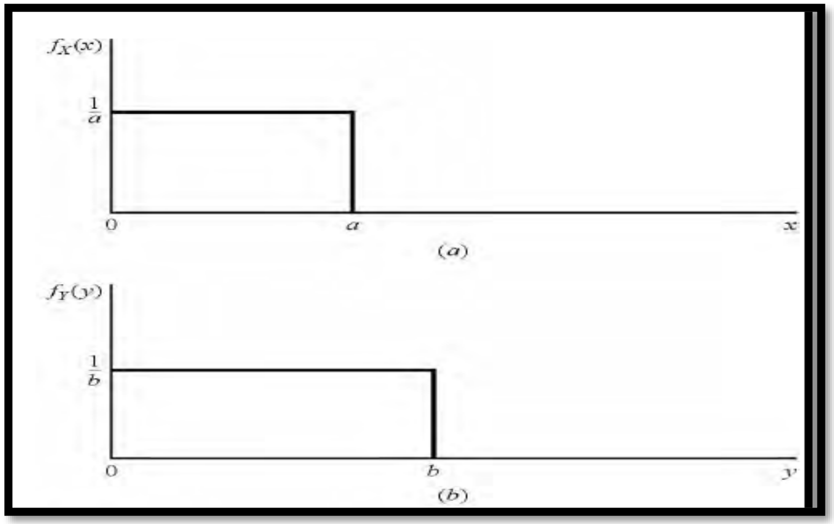
$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y) f_X(w-y)\,dy = f_X(x) * f_Y(y)$$

○ **Example**: Find the density of $W = X + Y$ where

$$f_X(x) = \frac{1}{a}[u(x) - u(x-a)]$$

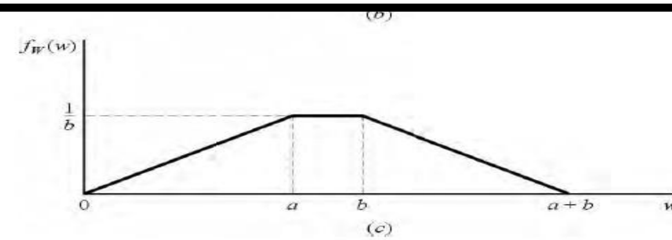
$$f_Y(y) = \frac{1}{b}[u(y) - u(y-b)]$$

with $0 < a < b$



- **Solution**:

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y) f_X(w-y)\,dy$$

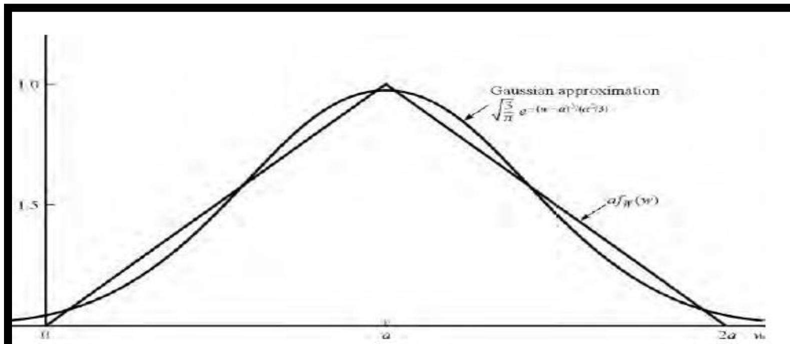


Central Limit Theorem

✓ The **central limit theorem** says that the probability distribution function of the **sum of a large number of random variables** approaches a Gaussian distribution

- **Example:**

$$f_X(x) = \frac{1}{a}[u(x) - u(x-a)]$$
$$f_Y(y) = \frac{1}{a}[u(y) - u(y-a)]$$



Examples

○ **Example**: If the joint probability density of **X** and **Y** is given by

$$f(x,y) = \begin{cases} x+y & \text{for } 0<x<1, 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the **joint distribution function** of these two random variables.

- **Solution**
  - ❋If either **x<0** or **y<0**, it follows immediately that **F(x, y) = 0**
  - ❋For **0<x<1** and **0<y<1** (Region I of Figure), we get

$$F(x,y) = \int_0^y \int_0^x (s+t)\, ds\, dt = \frac{1}{2}xy(x+y)$$

  - ❋For **x>1** and **0<y<1** (Region II of Figure), we get

$$F(x,y) = \int_0^y \int_0^1 (s+t)\, ds\, dt = \frac{1}{2}y(y+1)$$

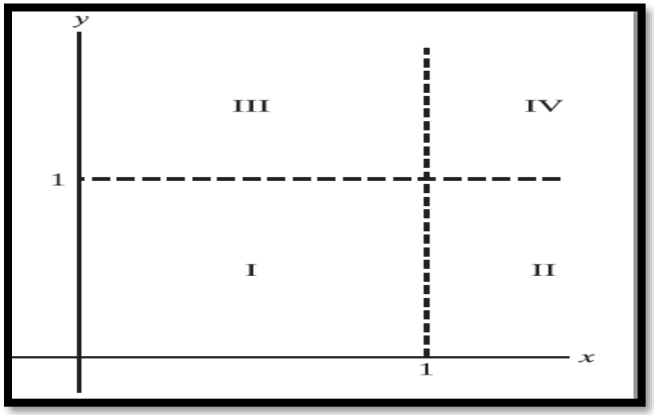


Figure: Diagram for Example

❈For **0<x<1** and **y>1** (Region III of Figure), we get

$$F(x,y)=\int_0^1\int_0^x (s+t)\,ds\,dt=\frac{1}{2}x(x+1)$$

❈for **x>1** and **y>1** (Region IV of Figure), we get

$$F(x,y)=\int_0^1\int_0^1 (s+t)\,ds\,dt=1$$

$$F(x,y)=\begin{cases} 0 & \text{for } x\leqq 0 \text{ or } y\leqq 0 \\ \frac{1}{2}xy(x+y) & \text{for } 0<x<1, 0<y<1 \\ \frac{1}{2}y(y+1) & \text{for } x\geqq 1, 0<y<1 \\ \frac{1}{2}x(x+1) & \text{for } 0<x<1, y\geqq 1 \\ 1 & \text{for } x\geqq 1, y\geqq 1 \end{cases}$$

- **Example**: Given the joint probability density

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) & \text{for } 0<x<1, 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the marginal densities of **X** and **Y.**

- **Solution**

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{2}{3}(x+2y)\,dy = \frac{2}{3}(x+1)$$

for **0<x<1** and **g(x) = 0** elsewhere

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{2}{3}(x+2y)\,dx = \frac{1}{3}(1+4y)$$

for **0<y<1** and **h(y) = 0** elsewhere.

- **Example:** Given the joint probability table, find the conditional distribution of **X** given **Y = 1**

		x		
		0	1	2
	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$
y	1	$\frac{2}{9}$	$\frac{1}{6}$	
	2	$\frac{1}{36}$		

- **Solution**

		x			
		0	1	2	
	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{7}{12}$
y	1	$\frac{2}{9}$	$\frac{1}{6}$		$\frac{7}{18}$
	2	$\frac{1}{36}$			$\frac{1}{36}$
		$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{12}$	

$$f(0|1) = \frac{\frac{2}{9}}{\frac{7}{18}} = \frac{4}{7}$$

$$f(1|1) = \frac{\frac{1}{6}}{\frac{7}{18}} = \frac{3}{7}$$

$$f(2|1) = \frac{0}{\frac{7}{18}} = 0$$